

# Asymptotic Concentration Behaviors of Linear Combinations of Weight Distributions on Random Linear Code Ensemble

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**Abstract**—Asymptotic concentration behaviors of linear combinations of weight distributions on the random linear code ensemble are presented. Many important properties of a binary linear code can be expressed as the form of a linear combination of weight distributions such as number of codewords, undetected error probability and upper bound on the maximum likelihood error probability. The key in this analysis is the covariance formula of weight distributions of the random linear code ensemble, which reveals the second-order statistics of a linear function of the weight distributions. Based on the covariance formula, several expressions of the *asymptotic concentration rate*, which indicate the speed of convergence to the average, are derived.

## I. INTRODUCTION

For a binary random code ensemble or a binary random linear code ensemble, the asymptotic behaviors of the first moment (expectation) of some properties of interest have been studied extensively. For example, the error exponent derived by Gallager [1] is a celebrated consequence of such a first-moment analysis. Recent advances in second-moment analysis on low-density parity check matrix ensembles [5], [6] have encouraged studies on the second-order behaviors (fluctuation from the average) of the macroscopic properties of an ensemble, which had previously attracted little attention.

In this paper, asymptotic concentration behaviors of linear combinations of weight distributions on the random linear code ensemble are presented. Many important properties of a binary linear code can be expressed as the form of a linear combination of weight distributions such as number of codewords, undetected error probability and upper bound on the maximum likelihood (ML) error probability. The key in this analysis is the covariance formula of weight distributions of the random linear code ensemble, which reveals the second-order statistics of a linear function of the weight distributions. Based on the covariance formula, several expressions of the *asymptotic concentration rate*, which indicate the speed of convergence to the average, are derived.

## II. PRELIMINARIES

### A. Ensemble, expectation, and covariance

Let  $\mathcal{G}$  be a set of binary  $m \times n$  matrices where  $m$  and  $n$  are positive integers. Suppose that probability  $P(H)$  is assigned for each matrix  $H$  in  $\mathcal{G}$ , where  $P(H)$  is a probability mass

function defined on  $\mathcal{G}$  such that  $\sum_{H \in \mathcal{G}} P(H) = 1$ , and  $\forall H \in \mathcal{G}, P(H) > 0$ . The pair  $\{\mathcal{G}, P(H)\}$  can be considered as an *ensemble of matrices*. Although it is an abuse of notation, for simplicity, we will not distinguish  $\{\mathcal{G}, P(H)\}$  from  $\mathcal{G}$ .

Let  $f(\cdot)$  be a real-valued function defined on  $\mathcal{G}$ , which can be considered as a *random variable*. The expectation of  $f(\cdot)$  with respect to the ensemble  $\mathcal{G}$  is defined by  $E_{\mathcal{G}}[f] \triangleq \sum_{H \in \mathcal{G}} P(H)f(H)$ . The variance of  $f(\cdot)$  is given by  $\text{VAR}_{\mathcal{G}}[f] \triangleq E_{\mathcal{G}}[f(H)^2] - E_{\mathcal{G}}[f(H)]^2$ . In a similar way, the covariance between two real-valued functions  $f(\cdot), g(\cdot)$  defined on  $\mathcal{G}$  is given by

$$\text{COV}_{\mathcal{G}}[f, g] \triangleq E_{\mathcal{G}}[fg] - E_{\mathcal{G}}[f]E_{\mathcal{G}}[g]. \quad (1)$$

Let  $\{g_1(\cdot), g_2(\cdot), \dots, g_n(\cdot)\}$  be a set of real-valued functions defined on  $\mathcal{G}$ , and let  $f(\cdot)$  be a linear combination of  $g_i(\cdot)$ :  $f(H) \triangleq \sum_{i=1}^n \phi_i g_i(H)$  for  $H \in \mathcal{G}$ , where  $\phi_i (i \in [1, n])$  are real values. The notation  $[a, b]$  denotes the set of consecutive integers from  $a$  to  $b$ . It is easy to show that the variance of  $f(\cdot)$  is given by

$$\text{VAR}_{\mathcal{G}}[f] = \sum_{i=1}^n \sum_{j=1}^n \phi_i \phi_j \text{COV}_{\mathcal{G}}[g_i, g_j], \quad (2)$$

e.g., see [7] for details.

### B. Weight distribution

The weight distributions  $\{A_1(\cdot), \dots, A_n(\cdot)\}$ , which can be considered as a set of real-valued functions defined on  $\mathcal{G}$ , is defined by

$$A_w(H) \triangleq \sum_{\mathbf{x} \in Z^{(n, w)}} I[H\mathbf{x} = \mathbf{0}^m], \quad w \in [0, n], \quad (3)$$

for any  $H \in \mathcal{G}$ , where  $Z^{(n, w)}$  denotes the set of all binary  $n$ -tuples with weight  $w$ . The function  $I[\cdot]$  is the indicator function such that  $I[\text{condition}] = 1$  if *condition* is true; otherwise, it gives 0. In the present paper, symbol shown in bold, such as  $\mathbf{x}$ , denote column vectors.

Let  $C(H)$  be the binary linear code defined based on  $H$ , namely,  $C(H) \triangleq \{\mathbf{x} \in F_2^n : H\mathbf{x} = \mathbf{0}^m\}$ , where  $F_2$  denotes the binary Galois field. Many properties of  $C(H)$  of interest can be represented by a linear combination of the weight distributions  $\{A_w(\cdot)\}_{w=1}^n$ . Let  $F(\cdot)$  be such a property of  $C(H)$ , which is expressed as  $F(H) \triangleq \sum_{w=1}^n \Phi_w A_w(H)$  for any  $H \in \mathcal{G}$ , where  $\Phi_w (w \in [0, n])$  are real values.

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For example, the undetected error probability of  $C(H)$  can be expressed as a linear combination of the weight distributions of  $C(H)$  when it is used as an error detection code for a binary symmetric channel (BSC). The expression is given by  $F(H) = \sum_{w=1}^n A_w(H) \epsilon^w (1-\epsilon)^{n-w}$ , where  $\epsilon$  is the crossover probability of the BSC.

In this setting, the property  $F(\cdot)$  can be regarded as a random variable that takes a real value. It is natural to study its statistics such as expectation, variance for a given ensemble of binary matrices.

### C. Random linear code ensemble

In the present paper, we deal with an ensemble of binary matrices, which is called the *random linear code ensemble*.

**Definition 1:** The random linear code ensemble  $\mathcal{R}_{n,m}$  contains all binary  $m \times n$  matrices. Equal probability  $P(H) = 1/2^{nm}$  is assigned for each matrices in  $\mathcal{R}_{n,m}$ .  $\square$

Note that although the random linear code ensemble is actually an ensemble of matrices, it is regarded herein as an ensemble of binary linear codes.

The expectation of weight distributions of random ensemble is known [2] to be  $E_{\mathcal{R}_{n,m}}[A_w] = \binom{n}{w} 2^{-m}$  for  $n \geq 1$ . The next theorem provides a closed formula of the covariance of weight distributions over the random linear code ensemble.

**Theorem 1:** Assume a random ensemble  $\mathcal{R}_{n,m}$ . The covariance of  $A_{w_1}(\cdot)$  and  $A_{w_2}(\cdot)$  is given by

$$\begin{aligned} \text{COV}_{\mathcal{R}_{n,m}}[A_{w_1}, A_{w_2}] \\ = \begin{cases} 0, & 0 < w_1, w_2 \leq n, w_1 \neq w_2 \\ (1 - 2^{-m}) 2^{-m} \binom{n}{w}, & 0 < w_1 = w_2 \leq n. \end{cases} \end{aligned} \quad (4)$$

(Proof) The proof is given in Appendix.  $\square$

The variance of the weight distributions of the random linear code ensemble has already been shown in [4]. Thus, the new contribution of this theorem is the case in which  $\text{COV}_{\mathcal{R}_{n,m}}(A_{w_1}, A_{w_2}) = 0$  when  $w_1 \neq w_2$ . This theorem implies that the pair of random variables  $A_{w_1}$  and  $A_{w_2}$  ( $w_1 \neq w_2$ ) is *pairwise independent*<sup>1</sup>.

## III. FORMULAS ON ASYMPTOTIC CONCENTRATION RATE

### A. Asymptotic behaviors of expectation

**Definition 2:** Let  $\mathcal{G}_n$  be an ensemble of binary  $(1-R)n \times n$  matrices. The parameter  $R$ , called the design rate, is a real value in the range of  $0 < R < 1$ . Suppose that  $f(\cdot)$  is a real-valued function defined on  $\mathcal{G}$ . The *asymptotic exponent* of  $E_{\mathcal{G}}[f]$  is given by

$$\xi \triangleq \lim_{n \rightarrow \infty} \frac{1}{n} \log E_{\mathcal{G}_n}[f] \quad (5)$$

if the limit exists.  $\square$

Namely, asymptotically,  $E_{\mathcal{G}}[f]$  behaves like  $E_{\mathcal{G}}[f(H)] \doteq 2^{\xi n}$  where the notation  $a_n \doteq b_n$  means that

$$\lim_{n \rightarrow \infty} (1/n) \log a_n = \lim_{n \rightarrow \infty} (1/n) \log b_n.$$

In the present paper, a logarithm of base 2 is denoted by  $\log$ .

<sup>1</sup>Note that the set of random variables  $\{A_1, \dots, A_n\}$  are not mutually independent because  $\sum_{w=1}^n A_w(H) \geq 2^{n-m} - 1$  holds for any instance  $H$  in  $\mathcal{R}_{n,m}$ .

In the case of the random linear code ensemble, it has been reported [2] that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log E_{\mathcal{R}_{n,(1-R)n}}[A_{\theta n}] = H(\theta) - (1-R), \quad (6)$$

holds for  $0 < \theta \leq 1$ , where  $H(\cdot)$  is the binary entropy function defined by  $H(x) \triangleq -x \log x - (1-x) \log(1-x)$ . The parameter  $\theta$  is called the *normalized weight*.

### B. Asymptotic concentration rate

As the size of the matrix goes to infinity, the value of  $f(\cdot)$  is often sharply concentrated around its expectation. The asymptotic concentration rate is defined as follows.

**Definition 3:** Let  $\mathcal{G}_n$  be an ensemble of binary  $(1-R)n \times n$  matrices, where  $R$  is a real value in the range of  $0 < R < 1$ . For a real-valued function  $f(\cdot)$  defined on  $\mathcal{G}_n$ , the asymptotic concentration rate (abbreviated as ACR) of  $f(\cdot)$  is defined by

$$\eta \triangleq \lim_{n \rightarrow \infty} \frac{1}{n} \log \left( \frac{\text{VAR}_{\mathcal{G}_n}[f]}{E_{\mathcal{G}_n}[f]^2} \right). \quad (7)$$

if the limit exists.  $\square$

The following lemma explains the importance of the asymptotic concentration rate.

**Lemma 1:** Let  $\eta$  be the asymptotic concentration rate of  $f(\cdot)$ . For any positive real number  $\alpha$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \Pr \left[ \frac{f(H)}{E_{\mathcal{G}_n}[f]} \notin (1-\alpha, 1+\alpha) \right] \leq \eta \quad (8)$$

holds if  $E_{\mathcal{G}_n}[f] > 0$  for any sufficiently large  $n$ .

(Proof) Based on the Chebyshev inequality, the inequality

$$\Pr \left[ |f(H) - E_{\mathcal{G}_n}[f]| > c \sqrt{\text{VAR}_{\mathcal{G}_n}[f]} \right] \leq \frac{1}{c^2} \quad (9)$$

holds for any real number  $c > 0$ . Suppose that  $c$  is given by

$$c = \frac{\alpha E_{\mathcal{G}_n}[f]}{\sqrt{\text{VAR}_{\mathcal{G}_n}[f]}}. \quad (10)$$

where  $\alpha$  is a positive real number. From the assumption  $E_{\mathcal{G}_n}[f] > 0$ , it is easy to verify that  $c$  becomes positive. Substituting (10) into (9), we have

$$\Pr [|f(H) - E_{\mathcal{G}_n}[f]| > \alpha E_{\mathcal{G}_n}[f]] \leq \frac{\text{VAR}_{\mathcal{G}_n}[f]}{\alpha^2 E_{\mathcal{G}_n}[f]^2}. \quad (11)$$

Due to the assumption  $E_{\mathcal{G}}[f(H)] > 0$ , the above inequality can be rewritten in the following form:

$$\Pr \left[ \frac{f(H)}{E_{\mathcal{G}_n}[f]} \notin (1-\alpha, 1+\alpha) \right] \leq \frac{\text{VAR}_{\mathcal{G}_n}[f]}{\alpha^2 E_{\mathcal{G}_n}[f]^2}. \quad (12)$$

Considering the asymptotic exponent of the above equation, we obtain the claim of the lemma.  $\square$

From the asymptotic concentration rate, we can clarify the probabilistic convergence behavior of  $f(\cdot)$ . If  $\eta < 0$  holds,  $f(H)/E_{\mathcal{G}_n}[f]$  converges to 1 in probability as  $n$  goes to infinity. This means that  $\eta < 0$  is a sufficient condition of the convergence in probability. The asymptotic concentration rate indicates the speed of this convergence

*Example 1:* The variance of the weight distributions of the random linear code ensemble is given by

$$\text{VAR}_{\mathcal{R}_{n,(1-R)n}}[A_{\theta n}] = (1 - 2^{-(1-R)n})2^{-(1-R)n} \binom{n}{\theta n}. \quad (13)$$

Therefore, the asymptotic exponent of the variance becomes

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \text{VAR}_{\mathcal{R}_{n,(1-R)n}}[A_{\theta n}] = H(\theta) - (1 - R). \quad (14)$$

From this exponent, we immediately have the asymptotic concentration rate of the weight distribution:

$$\begin{aligned} \eta &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{\text{VAR}_{\mathcal{R}_{n,(1-R)n}}[A_{\theta n}]}{\mathbb{E}_{\mathcal{R}_{n,(1-R)n}}[A_{\theta n}]^2} \\ &= H(\theta) - (1 - R) - 2(H(\theta) - (1 - R)) \\ &= 1 - R - H(\theta). \end{aligned} \quad (15)$$

Let the minimum root of equation  $1 - R - H(\theta) = 0$  be  $\theta_{GV}$ , which is called the *relative Gilbert-Varshamov (GV) distance*. Since  $\eta < 0$  holds in the range  $\theta_{GV} < \theta < 1 - \theta_{GV}$ ,  $A_{\theta n}(H)/\mathbb{E}_{\mathcal{R}_{n,(1-R)n}}[A_{\theta n}]$  converges to 1 in probability as  $n$  goes to infinity [3].  $\square$

### C. ACR of a linear combination of weight distributions

The goal of the present paper is to observe the asymptotic behavior of the variance of the linear combination defined in (16) of the weight distributions:

$$F(H) = \sum_{w=1}^n \Phi_w A_w(H). \quad (16)$$

The next theorem gives the asymptotic concentration rate of  $F(H)$ .

*Theorem 2:* Let  $\mathcal{G}_n$  be an ensemble of binary  $(1 - R)n \times n$  matrices, which have the following asymptotic first- and second-order behaviors:

$$\mathbb{E}_{\mathcal{G}_n}[A_{\theta n}] \doteq 2^{n(H(\theta) + q(\theta))}, \quad (17)$$

$$\text{COV}_{\mathcal{G}_n}[A_{\theta_1 n}, A_{\theta_2 n}] \doteq 2^{n\gamma(\theta_1, \theta_2)}. \quad (18)$$

The asymptotic concentration rate of  $F(\cdot)$  defined in (16) is given by

$$\begin{aligned} \eta &= \sup_{0 < \theta_1 \leq 1} \sup_{0 < \theta_2 \leq 1} [\phi(\theta_1) + \phi(\theta_2) + \gamma(\theta_1, \theta_2)] \\ &\quad - 2 \sup_{0 < \theta \leq 1} [\phi(\theta) + H(\theta) + q(\theta)], \end{aligned} \quad (19)$$

where  $\phi(\theta)$  is defined by

$$\phi(\theta) \triangleq \lim_{n \rightarrow \infty} \frac{1}{n} \log \Phi_{\theta n}. \quad (20)$$

(Proof) It is easy to verify that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}_{\mathcal{G}_n}[F(H)] = \sup_{0 < \theta \leq 1} [\phi(\theta) + H(\theta) + q(\theta)] \quad (21)$$

holds. Using Eq.(2), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \log \text{VAR}_{\mathcal{G}_n}[F] \\ = \sup_{0 < \theta_1 \leq 1} \sup_{0 < \theta_2 \leq 1} [\phi(\theta_1) + \phi(\theta_2) + \gamma(\theta_1, \theta_2)]. \end{aligned} \quad (22)$$

Substituting (21) and (22) into the definition of the ACR, the theorem is proven.  $\square$

The next corollary is a special case of the above theorem for the random linear code ensemble.

*Corollary 1:* The ACR of  $F(\cdot)$  defined in (16) over the random linear code ensemble  $\mathcal{R}_{n,(1-R)n}$  is given by

$$\eta = \sup_{0 < \theta \leq 1} [2\phi(\theta) + H(\theta)] - \sup_{0 < \theta \leq 1} [2\phi(\theta) + 2H(\theta)] + 1 - R, \quad (23)$$

where  $\phi$  is given in (20).

(Proof) In the case of the random ensemble,  $q(\theta)$  is given by  $q(\theta) = -(1 - R)$  for  $0 < \theta \leq 1$ . From Theorem 1, we can derive the exponent of the covariance  $\gamma(\theta_1, \theta_2)$ , which is given by

$$\gamma(\theta_1, \theta_2) = \begin{cases} -\infty, & \theta_1 \neq \theta_2 \\ H(\theta) - (1 - R), & \theta_1 = \theta_2, \end{cases} \quad (24)$$

where  $0 < \theta_1, \theta_2 \leq 1$ . Plugging these functions into the formula in Theorem 2, we obtain the claim of the corollary.  $\square$

*Example 2:* In this example, we will discuss the number of codewords in  $C(H)$ . Let us define  $M(H) \triangleq 1 + \sum_{w=1}^n A_w(H)$ , which is the number of codewords of  $C(H)$ . In this case, we can see that  $\Phi_w = 1$  holds for  $1 \leq w \leq n$ . The asymptotic exponent of  $M(H)$  is given by

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}_{\mathcal{R}_{n,m}}[M] &= \sup_{0 < \theta \leq 1} [H(\theta)] - (1 - R) \\ &= R. \end{aligned} \quad (25)$$

From the definition of  $M(H)$ , we immediately have  $\phi(\theta) = 0$ ,  $0 < \theta \leq 1$ . Using Corollary 1, we obtain

$$\begin{aligned} \eta &= \sup_{0 < \theta \leq 1} [H(\theta)] - \sup_{0 < \theta \leq 1} [2H(\theta)] + 1 - R \\ &= -R. \end{aligned} \quad (26)$$

Since  $R$  is a positive real number,  $M(H)/\mathbb{E}_{\mathcal{R}_{n,m}}[M]$  converges to 1 in probability for any  $R > 0$ .  $\square$

In some cases, the asymptotic concentration rate can be written in a closed form without an optimization process required in Corollary 1.

*Theorem 3:* Assume the random linear code ensemble with design rate  $R$ . Let  $K_1, K_2$  be real positive constants that do not depend on  $n$ . If  $\Phi_w$  is expressed as  $\Phi_w = K_1^w K_2^{n-w}$ , then the ACR of  $F(H) = \sum_{w=1}^n \Phi_w A_w(H)$  is given by

$$\eta = \log \frac{K_1^2 + K_2^2}{(K_1 + K_2)^2} + 1 - R. \quad (27)$$

(Proof) Using Theorem 1 and the binomial theorem, we have

$$\begin{aligned} \text{VAR}_{\mathcal{R}_{n,(1-R)n}}[F] &= \sum_{w_1=1}^n \sum_{w_2=1}^n (K_1^{w_1+w_2} K_2^{2n-w_1-w_2}) \text{COV}_{\mathcal{R}_{n,(1-R)n}}[A_{w_1}, A_{w_2}] \\ &= \sum_{w=1}^n (K_1^{2w} K_2^{2n-2w}) (1 - 2^{-m}) 2^{-m} \binom{n}{w} \\ &= (1 - 2^{-m}) 2^{-m} \left( \sum_{w=0}^n \binom{n}{w} (K_1^2)^w (K_2^2)^{n-w} \right) \end{aligned}$$

$$\begin{aligned}
& - (1 - 2^{-m})2^{-m}K_2^{2n} \\
& = (1 - 2^{-m})2^{-m} (K_1^2 + K_2^2)^n - (1 - 2^{-m})2^{-m}K_2^{2n}. \quad (28)
\end{aligned}$$

Thus, the asymptotic exponent of  $\text{VAR}_{\mathcal{R}_{n,(1-R)n}}[F]$  is given by

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \text{VAR}_{\mathcal{R}_{n,(1-R)n}}[F] = \log(K_1^2 + K_2^2) - (1 - R). \quad (29)$$

In a similar way,  $E_{\mathcal{R}_{n,(1-R)n}}[F]$  can be rewritten as follows:

$$\begin{aligned}
E_{\mathcal{R}_{n,(1-R)n}}[F] &= \sum_{w=1}^n (K_1^w K_2^{n-w}) E_{\mathcal{R}_{n,(1-R)n}}[A_w] \\
&= 2^{-m} \left( \sum_{w=0}^n (K_1^w K_2^{n-w}) \binom{n}{w} \right) - 2^{-m} K_2^n \\
&= 2^{-m} (K_1 + K_2)^n - 2^{-m} K_2^n. \quad (30)
\end{aligned}$$

This leads to the exponent of the expectation:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log E_{\mathcal{R}_{n,(1-R)n}}[F] = \log(K_1 + K_2) - (1 - R). \quad (31)$$

Substituting the above two equations into the definition of the ACR, we have the claim of the theorem.  $\square$

*Example 3:* Assume the binary symmetric channel with crossover probability  $\epsilon$ . The undetected error probability of  $C(H)$  is given by  $P_U(H) = \sum_{w=1}^n A_w(H) \epsilon^w \epsilon^{n-w}$ . In this case, the error exponent becomes

$$\lim_{n \rightarrow \infty} \frac{-1}{n} E_{\mathcal{R}_{n,(1-R)n}}[P_U] = 1 - R. \quad (32)$$

Since  $\Phi_w = \epsilon^w \epsilon^{n-w}$  has the form stated in Theorem 3 (i.e.,  $K_1 = \epsilon$ ,  $K_2 = 1 - \epsilon$ ), we can apply Theorem 3 and obtain  $\eta = \log(\epsilon^2 + (1 - \epsilon)^2) + 1 - R$ . This results suggests the existence of the convergence threshold  $\epsilon^*$  for given  $R$  such that  $\epsilon^*$  separates the concentration regime and the non-concentration regime of  $\epsilon$ . The root of  $\log(\epsilon^2 + (1 - \epsilon)^2) + 1 - R = 0$  becomes an upper bound of  $\epsilon^*$ . Let  $\epsilon'$  be the root of the equation  $\log(\epsilon'^2 + (1 - \epsilon')^2) + 1 - R = 0$ . Table I presents some values of  $\epsilon'$  for  $0.1 \leq R \leq 0.9$ . When  $\epsilon > \epsilon'$ , we have  $\log(\epsilon'^2 + (1 - \epsilon')^2) + 1 - R < 0$ . In such a region,  $P_U(\cdot)$  concentrates around its average value in the limit as  $n$  tends to infinity.

TABLE I  
ROOTS OF  $\log(\epsilon^2 + (1 - \epsilon)^2) + 1 - R = 0$

$R$	$\epsilon'$
0.1	0.366047
0.2	0.307193
0.3	0.259613
0.4	0.217375
0.5	0.178203
0.6	0.140933
0.7	0.104872
0.8	0.069564
0.9	0.034687

#### IV. ACR OF THE UPPER BOUND OF ML ERROR PROBABILITY

##### A. Bhattacharya bound

In the following discussion, the binary symmetric channel with crossover probability  $\epsilon$  is assumed for simplicity. Assume

that ML decoding is used in a decoder. For a binary  $m \times n$  parity check matrix  $H$ , the block error probability  $P_e(H)$  can be upper bounded as follows:

$$P_e(H) \leq \sum_{w=1}^n A_w(H) D^w,$$

where  $D$  is called the Bhattacharya parameter and is defined as

$$D \triangleq 2\sqrt{\epsilon(1 - \epsilon)}.$$

The upper bound is called the *Bhattacharya bound* [1] and has the form of a linear combination of weight distributions. Let us define  $B(H) \triangleq \sum_{w=1}^n A_w(H) D^w$ . It is expected that the statistics of  $B(H)$  reflects the asymptotic behavior of actual ML probability of an ensemble.

We first derive the asymptotic expression of the error exponent of the Bhattacharya bound in the case of the random linear code ensemble. The expectation of  $B(H)$  has the following closed form expression:

$$\begin{aligned}
E_{\mathcal{R}_{n,(1-R)n}}[B] &= \sum_{w=1}^n E_{\mathcal{R}_{n,(1-R)n}}[A_w(H)] D^w \\
&= \sum_{w=1}^n \binom{n}{w} 2^{-(1-R)n} \left( 2\sqrt{\epsilon(1 - \epsilon)} \right)^w \\
&= 2^{-(1-R)n} (2\sqrt{\epsilon(1 - \epsilon)} + 1)^n - 2^{-(1-R)n}.
\end{aligned}$$

Thus, the error exponent of  $E_{\mathcal{R}_{n,(1-R)n}}[B]$  is given by

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{-1}{n} \log E_{\mathcal{R}_{n,(1-R)n}}[B] \\
= 1 - R - \log \left( 2\sqrt{\epsilon(1 - \epsilon)} + 1 \right). \quad (33)
\end{aligned}$$

This is a part of the error exponent function derived by Gallager [1] (see also [3]) in the low-rate regime<sup>2</sup>. Namely, the Bhattacharya bound corresponds to the upper bound due to Gallager with the parameter  $\rho = 1$  [1].

In the following, we will examine the asymptotic concentration rate of the Bhattacharya bound.

*Corollary 2:* The ACR of  $B(H)$  is given by

$$\eta = \log \left( \frac{4\epsilon(\epsilon - 1) + 1}{(2\sqrt{\epsilon(1 - \epsilon)} + 1)^2} \right) + 1 - R. \quad (34)$$

(Proof) By letting  $K_1 = D$  and  $K_2 = 1$  and using Theorem 3, we obtain  $\eta = \log((D^2 + 1)/(D + 1)^2) + 1 - R$ . Substituting  $D = 2\sqrt{\epsilon(1 - \epsilon)}$  into this equation, the corollary is proven.  $\square$

##### B. Expurgated bound

We here consider the expurgated ensemble  $\mathcal{R}_{n,(1-R)n}^*$ , which can be obtained from  $\mathcal{R}_{n,(1-R)n}$  by expurgating parity check matrices with  $A_{\theta n}(H) \neq 0$  for  $0 < \theta < \theta_{GV}$ ,  $1 - \theta_{GV} < \theta \leq 1$ . The asymptotic growth rate of the weight distributions is the same for the original and expurgated ensembles when  $\theta_{GV} \leq \theta \leq 1 - \theta_{GV}$ . However,  $q(\theta)$  becomes  $-\infty$  when

<sup>2</sup>It has been reported that this exponent is asymptotically tight if  $R_x \leq R \leq R_{crit}$  [3].

$0 < \theta < \theta_{GV}$ ,  $1 - \theta_{GV} < \theta \leq 1$  in the case of the expurgated ensemble.

The error exponent of  $E_{\mathcal{R}^*_{n,(1-R)n}}[B]$  is given by

$$\lim_{n \rightarrow \infty} \frac{-1}{n} \log E_{\mathcal{R}^*_{n,(1-R)n}}[B] = \min_{\theta_{GV} \leq \theta \leq 1 - \theta_{GV}} \{1 - R - H(\theta) - \theta \log(2\sqrt{\epsilon(1-\epsilon)})\}.$$

If  $\theta_{crit} \geq \theta_{GV}$ , the minimum in the above equation is attained at  $\theta = \theta_{crit}$ , where

$$\theta_{crit} \triangleq \frac{2\sqrt{\epsilon(1-\epsilon)}}{1 + 2\sqrt{\epsilon(1-\epsilon)}}. \quad (35)$$

In this case, the exponent coincides with the exponent given in Eq.(33). Otherwise, ( $\theta_{crit} < \theta_{GV}$ ), the minimum occurs at  $\theta = \theta_{GV}$ . Therefore, we have

$$\lim_{n \rightarrow \infty} \frac{-1}{n} \log E_{\mathcal{R}^*_{n,(1-R)n}}[B] = -\theta_{GV} \log(2\sqrt{\epsilon(1-\epsilon)}). \quad (36)$$

if  $\theta_{crit} < \theta_{GV}$ . This exponent corresponds to the usual *expurgated exponent* for the BSC case (see also the discussion in [3]). The next corollary states the ACR of the upper bound of ML error probability in the case of  $\theta_{crit} < \theta_{GV}$ :

*Corollary 3:* If  $\theta_{crit} < \theta_{GV}$ , the ACR is given by  $\eta = 0$ . (Proof) Since the expurgated ensemble can be obtained from the original ensemble by removing a sub-exponential number of matrices, the exponent of the variance, i.e.,  $\gamma(\theta_1, \theta_2)$ , takes the same values for the original and expurgated ensembles if  $\theta_{GV} \leq \theta_1, \theta_2 \leq 1 - \theta_{GV}$ . From Theorem 2, we have

$$\eta = \max_{\theta_{GV} \leq \theta \leq 1 - \theta_{GV}} [H(\theta) + 2\theta \log(2\sqrt{\epsilon(1-\epsilon)})] - \max_{\theta_{GV} \leq \theta \leq 1 - \theta_{GV}} [2H(\theta) + 2\theta \log(2\sqrt{\epsilon(1-\epsilon)})] + 1 - R$$

because  $q(\theta) = -\infty$  for  $\theta < \theta_{GV}$  in the case of the expurgated ensemble. From the assumption  $\theta_{crit} < \theta_{GV}$ ,  $2H(\theta) + 2\theta \log(2\sqrt{\epsilon(1-\epsilon)})$  is maximized at  $\theta = \theta_{GV}$ . Note that  $-H(\theta_{GV}) + 1 - R = 0$  holds. Moreover,  $H(\theta) + 2\theta \log(2\sqrt{\epsilon(1-\epsilon)})$  is also maximized at  $\theta = \theta_{GV}$ .  $\square$

## APPENDIX

1) *Preparation of the proof of Theorem 1:* The second moment of the weight distribution for a given ensemble  $\mathcal{G}$  is given by

$$\begin{aligned} E_{\mathcal{G}}[A_{w_1} A_{w_2}] &= E_{\mathcal{G}} \left[ \sum_{\mathbf{x} \in Z^{(n, w_1)}} \sum_{\mathbf{y} \in Z^{(n, w_2)}} I[H\mathbf{x} = \mathbf{0}^m, H\mathbf{y} = \mathbf{0}^m] \right] \\ &= \sum_{\mathbf{x} \in Z^{(n, w_1)}} \sum_{\mathbf{y} \in Z^{(n, w_2)}} E_{\mathcal{G}}[I[H\mathbf{x} = \mathbf{0}^m, H\mathbf{y} = \mathbf{0}^m]]. \end{aligned} \quad (37)$$

For the case in which  $\mathcal{G} = \mathcal{R}_{n,m}$ , we obtain

$$\begin{aligned} E_{\mathcal{R}_{n,m}}[A_{w_1} A_{w_2}] &= \sum_{\mathbf{x} \in Z^{(n, w_1)}} \sum_{\mathbf{y} \in Z^{(n, w_2)}} \frac{\#\{H : H\mathbf{x} = \mathbf{0}^m, H\mathbf{y} = \mathbf{0}^m\}}{2^{mn}} \end{aligned} \quad (38)$$

Here, we encounter a problem of counting the matrices that satisfy both  $H\mathbf{x} = \mathbf{0}^m$  and  $H\mathbf{y} = \mathbf{0}^m$ . Before solving this counting problem, we first introduce some notation.

Suppose that  $w_1 > 0$  and  $w_2 > 0$ . For a given pair  $(\mathbf{x}, \mathbf{y}) \in Z^{(n, w_1)} \times Z^{(n, w_2)}$ , the index sets  $I_1, I_2, I_3$ , and  $I_4$  are defined as follows:  $I_1 \triangleq \{k \in [1, n] : x_k = 1, y_k = 0\}$ ,  $I_2 \triangleq \{k \in [1, n] : x_k = 1, y_k = 1\}$ ,  $I_3 \triangleq \{k \in [1, n] : x_k = 0, y_k = 1\}$ ,  $I_4 \triangleq \{k \in [1, n] : x_k = 0, y_k = 0\}$ , where  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  and  $\mathbf{y} = (y_1, y_2, \dots, y_n)$ . The size of each index set is denoted by  $i_k = \#I_k$  ( $k = 1, 2, 3, 4$ ). Let  $\mathbf{h} = (h_1, h_2, \dots, h_n)^t$  be a binary  $n$ -tuple (a row vector). The partial weight of  $\mathbf{h}$  corresponding to an index set  $I_k$  ( $k = 1, 2, 3, 4$ ) is denoted by  $w_k(\mathbf{h})$ , namely,  $w_k(\mathbf{h}) \triangleq \#\{j \in I_k : h_j = 1\}$ .

Since the index sets are mutually exclusive, the equation  $i_1 + i_2 + i_3 + i_4 = n$  holds and  $i_2$  can take the integer values in the following range:  $\max\{w_1 + w_2 - n, 0\} \leq i_2 \leq \min\{w_1, w_2\}$ . The size of each index set can be expressed as  $i_1 = w_1 - i_2$ ,  $i_3 = w_2 - i_2$ ,  $i_4 = n - (w_1 + w_2 - i_2)$ .

The next lemma forms the basis for the proof of Theorem 1.

*Lemma 2:* For any  $\mathbf{x} \in Z^{(n, w_1)}$  and  $\mathbf{y} \in Z^{(n, w_2)}$  ( $0 < w_1, w_2 \leq n$ ), the following equalities hold:

$$\#\{\mathbf{h} \in F_2^n : \mathbf{h}\mathbf{x} = 0, \mathbf{h}\mathbf{y} = 0\} = \begin{cases} 2^{n-2} & \mathbf{x} \neq \mathbf{y}, \\ 2^{n-1} & \mathbf{x} = \mathbf{y}. \end{cases} \quad (39)$$

(Proof) In the following, we prove the lemma for the conditions  $0 < w_1 \leq w_2 \leq n$ . The proof for the opposite case  $0 < w_2 \leq w_1 \leq n$  then follows immediately upon exchanging the variables  $w_2$  and  $w_1$  in the proof.

First, we will show that

$$\#\{\mathbf{h} \in F_2^n : \mathbf{h}\mathbf{x} = 0, \mathbf{h}\mathbf{y} = 0\} = 2^{n-2} \quad (40)$$

if  $0 < w_1 \leq w_2 \leq n$  and  $\mathbf{x} \neq \mathbf{y}$ . Let the support sets of  $\mathbf{x}$  and  $\mathbf{y}$  be  $S(\mathbf{x}) \triangleq \{i \in [1, n] : x_i = 1\}$  and  $S(\mathbf{y}) \triangleq \{i \in [1, n] : y_i = 1\}$ , respectively. The following three cases should be treated separately:

- Case (i):  $0 < i_2 < w_1$  (i.e.,  $S(\mathbf{x})$  and  $S(\mathbf{y})$  overlap but  $S(\mathbf{y})$  does not include  $S(\mathbf{x})$ .)
- Case (ii):  $i_2 = 0$  (i.e.,  $S(\mathbf{x})$  and  $S(\mathbf{y})$  do not overlap.)
- Case (iii):  $i_2 = w_1$  (i.e.,  $S(\mathbf{y})$  includes  $S(\mathbf{x})$ .)

First, we consider Case (i). From the assumption that  $0 < i_2 < w_1$ , it is clear that  $I_1 \neq \emptyset$  (because  $i_2 < w_1$ ),  $I_2 \neq \emptyset$  (because  $i_2 > 0$ ),  $I_3 \neq \emptyset$  (because  $w_2 \geq w_1 > i_2$ ). For any  $\mathbf{h} \in F_2^n$ , the equations  $\mathbf{h}\mathbf{x}^t = 0$  and  $\mathbf{h}\mathbf{y}^t = 0$  hold if and only if  $w_i(\mathbf{h})$  is even for  $i = 1, 2, 3$  or  $w_i(\mathbf{h})$  is odd for  $i = 1, 2, 3$ . Thus, the number of vectors satisfying the above condition is given by

$$N_{\mathbf{h}} = 2 \times 2^{i_1-1} \times 2^{i_2-1} \times 2^{i_3-1} \times 2^{i_4} = 2^{n-2}, \quad (41)$$

where  $N_{\mathbf{h}}$  is defined by  $N_{\mathbf{h}} \triangleq \#\{\mathbf{h} \in F_2^n : \mathbf{h}\mathbf{x}^t = 0, \mathbf{h}\mathbf{y}^t = 0\}$ . In the above derivation, we used the equalities:  $w_1 = i_1 + i_2$ ,  $w_2 = i_2 + i_3$ ,  $i_4 = n - (w_1 + w_2 - i_2)$ . Note that Eq. (41) (and Eqs. (42), (43), and (44) to be presented below) holds regardless of the size of  $I_4$  ( $i_4 = 0$  or  $i_4 > 0$ ).

We now consider Case (ii). For this case,  $I_1 \neq \emptyset$  (since  $w_1 > 0$ ),  $I_2 = \emptyset$  (since  $i_2 = 0$ ) and  $I_3 \neq \emptyset$  (since  $w_2 > 0$ ). The equalities  $\mathbf{h}\mathbf{x} = 0$  and  $\mathbf{h}\mathbf{y} = 0$  hold if and only if  $w_i(\mathbf{h})$  is even for  $i = 1, 3$  holds. The number of vectors satisfying the condition is given by

$$N_{\mathbf{h}} = 2^{i_1-1} \times 2^{i_3-1} \times 2^{i_4} = 2^{n-2}. \quad (42)$$

The final case is Case (iii). For this case,  $I_1 = \emptyset$  (since  $i_2 = w_1$ ),  $I_2 \neq \emptyset$  (since  $i_2 = w_1 > 0$ ) and  $I_3 \neq \emptyset$  (since  $\mathbf{x} \neq \mathbf{y}$  and  $w_1 \leq w_2$ ). These conditions lead to the condition:  $w_i(\mathbf{h})$  is even for  $i = 2, 3$  for  $\mathbf{h}\mathbf{x} = 0, \mathbf{h}\mathbf{y} = 0$ . Again,  $2^{n-2}$   $n$ -tuples satisfy the above condition, namely,

$$N_{\mathbf{h}} = 2^{i_2-1} \times 2^{i_3-1} \times 2^{i_4} = 2^{n-2}. \quad (43)$$

Combining the above results for Cases (i), (ii), and (iii), we obtain Eq. (40).

We then show that  $N_{\mathbf{h}} = 2^{n-1}$  holds if  $0 < w_1 = w_2 \leq n$  and  $\mathbf{x} = \mathbf{y}$ . For this case, we have  $I_1 = \emptyset, I_2 \neq \emptyset, I_3 = \emptyset$  (since  $\mathbf{x} = \mathbf{y}$ ). Thus, the equations  $\mathbf{h}\mathbf{x} = 0, \mathbf{h}\mathbf{y} = 0$  hold if and only if  $w_2(\mathbf{h})$  is even. The number of  $n$ -tuples satisfying the above condition is given by

$$N_{\mathbf{h}} = 2^{i_2-1} \times 2^{i_4} = 2^{n-1}. \quad (44)$$

The proof of this lemma is completed.  $\square$

2) *Proof of Theorem 1:* The proof of Theorem 1 consists of two parts. The first part corresponds to the case in which the covariance becomes zero. The second part corresponds to the case in which the covariance becomes non-zero.

We commence with the first part of the proof. Assume that  $0 < w_1, w_2 \leq n, \mathbf{x} \neq \mathbf{y}$ . From Lemma 2, we obtain

$$\begin{aligned} & \#\{H : H\mathbf{x} = 0^m, H\mathbf{y} = 0^m\} \\ &= \prod_{k=1}^m \#\{\mathbf{h} \in F_2^n : \mathbf{h}\mathbf{x} = 0, \mathbf{h}\mathbf{y} = 0\} \\ &= 2^{m(n-2)}. \end{aligned} \quad (45)$$

Substituting into (38), we obtain

$$\begin{aligned} \mathbb{E}_{\mathcal{R}_{n,m}}[A_{w_1}A_{w_2}] &= \sum_{\mathbf{x} \in Z^{(n,w_1)}} \sum_{\mathbf{y} \in Z^{(n,w_2)}} \frac{2^{m(n-2)}}{2^{mn}} \\ &= 2^{-2m} \binom{n}{w_1} \binom{n}{w_2} \\ &= \mathbb{E}_{\mathcal{R}_{n,m}}[A_{w_1}] \mathbb{E}_{\mathcal{R}_{n,m}}[A_{w_2}]. \end{aligned} \quad (46)$$

The last equality is equivalent to  $\text{COV}_{\mathcal{R}_{n,m}}[A_{w_1}, A_{w_2}] = 0$ .

We now consider the second part of the proof: Assume that  $\mathbf{x} = \mathbf{y}$ . From Lemma 2, we have  $\#\{H : H\mathbf{x}^t = \mathbf{0}, H\mathbf{y}^t = \mathbf{0}\} = 2^{m(n-1)}$ , and

$$\begin{aligned} \mathbb{E}_{\mathcal{R}_{n,m}}[A_w^2] &= \sum_{\mathbf{x} \in Z^{(n,w)}} \sum_{\mathbf{y} \in Z^{(n,w)}} \frac{I[\mathbf{x} = \mathbf{y}] 2^{m(n-1)}}{2^{mn}} \\ &+ \sum_{\mathbf{x} \in Z^{(n,w)}} \sum_{\mathbf{y} \in Z^{(n,w)}} \frac{I[\mathbf{x} \neq \mathbf{y}] 2^{m(n-2)}}{2^{mn}} \\ &= 2^{-m} \binom{n}{w} + 2^{-2m} \left( \binom{n}{w} \binom{n}{w} - \binom{n}{w} \right) \\ &= \mathbb{E}_{\mathcal{R}_{n,m}}[A_w]^2 + 2^{-m} \binom{n}{w} - 2^{-2m} \binom{n}{w}. \end{aligned} \quad (47)$$

The last equality is equivalent to  $\text{COV}_{\mathcal{R}_{n,m}}(A_w, A_w) = (1 - 2^{-m})2^{-m} \binom{n}{w}$ .  $\square$

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